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ADP012045

TITLE: On Calculating with Lower Order Chebyshev Splines

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TITLE: International Conference on Curves and Surfaces [4th], Saint-Malo, France, 1-7 July 1999. Proceedings, Volume 1. Curve and Surface Design

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On Calculating with Lower Order Chebyshev Splines

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Abstract. We develop a technique to calculate with Chebyshev Splines of orders 3 and 4, based on the known derivative formula for Chebyshev splines and an Oslo type algorithm. We assume that splines in the reduced system are simple enough to calculate. Local bases of Chebyshev splines of order 3 and 4 can thus be evaluated as positive linear combinations of less smooth Chebyshev B-splines. The coefficients in such linear combinations are discrete Chebyshev splines, normalized so as to make a partition of unity. There are a number of interesting special cases, such as Foley's ν-splines, Chebyshev polynomial splines (q-splines), and splines in tension which can be calculated stably by such formulæ.

§1. Introduction and Preliminaries

It is an important fact in the univariate polynomial spline theory that splines can be represented as linear combinations of compactly supported basis functions, which we can calculate in various ways by stable and fast numerical algorithms. This can be extended to some other well-known spaces of splines, such as trigonometric and hyperbolic splines, where nice three-term recurrences of de Boor-Cox type exist; this also applies to a less interesting case of Chebyshev splines with equal weights. These issues have been discussed in [12], where we can also find a negative result concerning the existence of such three-term recurrence relations in general. Polynomials, however, form an algebra, while in other cases multiplicative properties are replaced by other algebraic formulæ, such as addition formulæ for trigonometric/hyperbolic functions or similar identities.

We shall say that an interval $[a, b]$ is measurable with respect to the measure vector $d\boldsymbol{\sigma} := (d\sigma_2, \dots d\sigma_n)^{\mathrm{T}}$ if it is measurable with respect to the positive Lebesgue-Stieltjes measures $d\sigma_i$, $i = 2, \dots n$. Then for $x \in [a, b]$ we can define generalized powers (Chebyshev system) $\{1, u_2, \dots u_n\}$:

$$u_2(x) = \int_a^x d\sigma_2(t_2); \dots ; u_n(x) = \int_a^x d\sigma_2(t_2) \dots \int_a^{t_{n-1}} d\sigma_n(t_n). \qquad (1)$$

Curve and Surface Design: Saint-Malo 1999
Pierre-Jean Laurent, Paul Sablonnière, and Larry L. Schumaker (eds.), pp. 343–352.

ISBN 0-8265-1356-5.

If all of the measures $d\sigma_i$ are dominated by the Lebesgue measure, then they possess densities $\frac{1}{p_i}$, $i = 2, \ldots n$; if p_i are smooth, *i.e.*,$\frac{1}{p_i} := \frac{d\sigma_i}{dt} \in C^{n-i+1}$, then $\{1, u_2, \ldots u_n\}$ is called an Extended Complete Chebyshev System (ECC-system), referred to as ECT system in [11]). Further, we shall assume that such an integral representation has been found, and the measures for the Chebyshev space determined in such a way that we know the generalized powers explicitly. This may not always be easy, and the choice of measures may not be unique. It is known that Chebyshev B-splines that make a partition of unity exist in this general case, and even that abstract recurrences [5,1] resembling the polynomial ones exist. Almost nothing can be said about their numerical stability, at least not until we employ a special measure vector, whence the abstract construction gets difficult.

Recently, other techniques based on blossoming have been found for the Chebyshev splines [6], which are more promising so far as evaluation and numerical stability is involved. In case of polynomial weights, one obtains Chebyshev polynomial splines [7], though here again the underlying algebraic properties of polynomials are implicitly used.

In Section 2 we give some formulæ for Chebyshev systems of orders 3 and 4 which express locally supported splines (being piecewise in these spaces) as positive linear combinations of less smooth splines in the same space. We will see that the coefficients are related to integrals of splines in the reduced system, which is defined to be a Chebyshev system like (1) corresponding to the reduced measure vector, that is, for each $\delta \subset [a, b]$

$$d\boldsymbol{\sigma}^{(i)}(\delta) := (d\sigma_{i+2}(\delta), \ldots, d\sigma_n(\delta))^{\mathrm{T}} \in \mathbb{R}^{n-(i+1)}, \qquad i = 1, \ldots n-2.$$

If $\mathcal{S}(n, d\boldsymbol{\sigma}) := \operatorname{span}\{1, u_2, \ldots u_n\}$, then the generalized derivatives $L_{j,d\boldsymbol{\sigma}} := D_j \cdots D_1 D_0$, where

$$D_j f(x) := \lim_{\delta \to 0+} \frac{f(x+\delta) - f(x)}{\sigma_{j+1}(x+\delta) - \sigma_{j+1}(x)} \quad j = 1, \ldots, n-1, \tag{2}$$

are linear mappings $\mathcal{S}(n, d\boldsymbol{\sigma}) \to \mathcal{S}(n-j, d\boldsymbol{\sigma}^{(j)})$. For a partition $\Delta = \{x_i\}_{i=0}^{k+1}$ of an interval $[a, b]$, and a given multiplicity vector $\boldsymbol{m} = (n_1, \ldots, n_k)^T$, $(0 < n_i \leq n)$, we shall denote by $\{t_1 \ldots t_{2n+k}\}$ [11] an extended partition in the usual way (see Fig. 1 and Fig. 2):

$$\begin{gathered} t_1 = \cdots = t_n = x_0, \\ t_{n+k+1} = \cdots = t_{2n+k} = x_{k+1}, \\ t_{n+1} \leq \cdots \leq t_{n+k} = \underbrace{x_1, \ldots, x_1}_{n_1}, \ldots, \underbrace{x_k, \ldots, x_k}_{n_k}. \end{gathered}$$

$\mathcal{S}(n, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ is the spline space spanned by functions being piecewise in $\mathcal{S}(n, d\boldsymbol{\sigma})$ [11]. Chebyshev B-splines in $\mathcal{S}(n, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$ have compact support

$[t_i, t_{i+n}]$, and we shall henceforth assume that these are unique such splines such that

$$\sum_{i=1}^{n+K} T^n_{i,d\boldsymbol{\sigma}^{(j)}}(x) = 1; \quad j = 0, \dots n-2, \tag{3}$$

where $K := \sum_i n_i$. We aim to show that for lower order Chebyshev splines, one can express Chebyshev B-splines as linear combinations with positive coefficients of B-splines with multiple knots. Hopefully, these can be computed efficiently by some interpolating formula or otherwise. In Section 3 we give some examples of how the theory can be used to construct Chebyshev B-splines in some known polynomial cases, like weighted splines and q-splines, but also in the non-trivial case of some other useful Chebyshev splines. To achieve this, we need some technical results. One is the derivative formula, stating that for $x \in [a, b]$, and a multiplicity vector $\boldsymbol{m}$ whose components satisfy $n_i < n-1$ $(i = 1, \dots k)$, the derivative of a T-spline is

$$L_{1,d\boldsymbol{\sigma}} T^n_{i,d\boldsymbol{\sigma}}(x) = \frac{T^{n-1}_{i,d\boldsymbol{\sigma}^{(1)}}(x)}{C_{n-1}(i)} - \frac{T^{n-1}_{i+1 d\boldsymbol{\sigma}^{(1)}}(x)}{C_{n-1}(i+1)}, \tag{4}$$

where

$$C_{n-1}(i) := \int_{t_i}^{t_{i+n-1}} T^{n-1}_{i,d\boldsymbol{\sigma}^{(1)}} d\sigma_2. \tag{5}$$

For continuous $\boldsymbol{\sigma}$, the proof is similar to that in [9]; a somewhat longer proof involving only determinant identities exists, and relies only on the fundamental theorem of integral calculus [11]:

$$f(b) - f(a) = \int_a^b L_{1,d\boldsymbol{\sigma}} f(t) d\sigma_2(t), \tag{6}$$

which holds under very weak hypothesis on the measures.

In certain cases one may construct splines defined on triplets of knots (Lagrangian splines) by a generalized Taylor formula. We give a variant of the Taylor formula, amenable to generalization, that can be useful for lower order Chebyshev systems:

Lemma 1. *If f and $d\boldsymbol{\sigma} = (d\sigma_2, \dots, d\sigma_5)^{\mathrm{T}}$ are such that $L_i f := L_{i,d\boldsymbol{\sigma}} f$ exist and are measurable with respect to $d\sigma_{i+1}, (i = 1, \dots 4)$ on $[a, b]$, then*

$$\begin{aligned} f(x) =& f(a) + L_1 f(a) \int_a^x d\sigma_2(s_2) + L_2 f(a) \int_a^x d\sigma_2(s_2) \int_a^{s_2} d\sigma_3(s_3) \\ &+ L_3 f(a) \int_a^x d\sigma_2(s_2) \int_a^{s_2} d\sigma_3(s_3) \int_a^{s_3} d\sigma_4(s_4) \\ &+ \int_a^x d\sigma_2(s_2) \int_a^{s_2} d\sigma_3(s_3) \int_a^{s_3} d\sigma_4(s_4) \int_a^{s_4} L_4 f(s_5) d\sigma_5(s_5). \end{aligned}$$

Proof: The hypothesis and (6) enable us to use standard arguments of mathematical analysis in developing a Taylor series expansion

$$\begin{aligned} f(x) =& f(a) + \int_a^x (L_1 f(s_2) - L_1 f(a) + L_1 f(a)) d\sigma_2(s_2) \\ &= f(a) + L_1 f(a) \int_a^x d\sigma_2(s_2) + \text{etc.} \quad \square \end{aligned}$$

Lemma 1 is in fact a way of writing a Taylor expansion for L-splines, see [13], pp. 425-426.

§2. A Knot Insertion Algorithm

We shall henceforth assume that Chebyshev B-splines $T^3_{i,d\boldsymbol{\sigma}^{(1)}}$ can be evaluated at the knots in a numericaly stable way. This is a sound hypothesis, since by (4),

$$T^3_{i,d\boldsymbol{\sigma}}(x) = \frac{1}{C_2(i)} \int_{t_i}^x T^2_{i,d\boldsymbol{\sigma}^{(1)}} d\sigma_2 - \frac{1}{C_2(i+1)} \int_{t_{i+1}}^x T^2_{i+1,d\boldsymbol{\sigma}^{(1)}} d\sigma_2.$$

If we suppose that the multiplicity vector $\boldsymbol{m}$ is such that $T^3_{i,d\boldsymbol{\sigma}} \in C[a,b]$ (that is, $n_i \leq 2$), then we have

$$T^3_{i,d\boldsymbol{\sigma}}(t_{i+1}) = \frac{1}{C_2(i)} \int_{t_i}^{t_{i+1}} T^2_{i,d\boldsymbol{\sigma}^{(1)}} d\sigma_2,$$

and

$$T^3_{i,d\boldsymbol{\sigma}}(t_{i+2}) = 1 - \frac{1}{C_2(i+1)} \int_{t_{i+1}}^{t_{i+2}} T^2_{i+1,d\boldsymbol{\sigma}^{(1)}} d\sigma_2 = \frac{1}{C_2(i+1)} \int_{t_{i+2}}^{t_{i+3}} T^2_{i+1,d\boldsymbol{\sigma}^{(1)}} d\sigma_2.$$

Since the construction of two-interval supported "linear" splines is easy, stable evaluation of $T^3_{i,d\boldsymbol{\sigma}}(t_{i+1})$, $T^3_{i,d\boldsymbol{\sigma}}(t_{i+2})$ amounts to finding an integration formula, preferably of Gaussian type. The important thing is that in this case we do not have any dangerous subtractions potentially leading to large floating point errors, as in (4).

Theorem 2. *Let $T^3_{j,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3, \boldsymbol{m}, d\boldsymbol{\sigma}^{(1)}, \Delta)$ be a Chebyshev 3rd order B-spline associated with the multiplicity vector $\boldsymbol{m} = (1, \ldots 1)^T$, and let us assume that $\tilde{T}^3_{j,d\boldsymbol{\sigma}^{(1)}} \in \mathcal{S}(3, \tilde{\boldsymbol{m}}, d\boldsymbol{\sigma}^{(1)}), \Delta)$ are Chebyshev B-splines associated with the multiplicity vector $\tilde{\boldsymbol{m}} = (2, \ldots 2)^T$ (Fig. 1) on the same partition. If $\{t_1, \ldots t_{k+6}\}$ and $\{\tilde{t}_1, \ldots \tilde{t}_{2k+6}\}$ are the associated extended partitions, and r an index such that $t_j = \tilde{t}_r < \tilde{t}_{r+1}$, then for $j = 1, \ldots k+3$,*

$$T^3_{j,d\boldsymbol{\sigma}^{(1)}} = T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+1}) \tilde{T}^3_{r,d\boldsymbol{\sigma}^{(1)}} + \tilde{T}^3_{r+1,d\boldsymbol{\sigma}^{(1)}} + T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+2}) \tilde{T}^3_{r+2,d\boldsymbol{\sigma}^{(1)}}.$$

x_0	x_1	x_2	x_3	x_4	x_5	x_6
$t_1\,\tilde{t}_1$	t_4	t_5	t_6	t_7	t_8	$t_9\,\tilde{t}_{14}$
$t_2\,\tilde{t}_2$	$\tilde{t}_4$	$\tilde{t}_6$	$\tilde{t}_8$	$\tilde{t}_{10}$	$\tilde{t}_{12}$	$t_{10}\,\tilde{t}_{15}$
$t_3\,\tilde{t}_3$	$\tilde{t}_5$	$\tilde{t}_7$	$\tilde{t}_9$	$\tilde{t}_{11}$	$\tilde{t}_{13}$	$t_{11}\,\tilde{t}_{16}$

Fig. 1. Double knots.

Proof: Let $T_j^3 := T^3_{j,d\boldsymbol{\sigma}^{(1)}}$. Since $T_j^3(t_j) = T_j^3(t_{j+3}) = 0$ and the same holds for the first generalized derivatives, we conclude that two out of three coefficients representing T_j^3 on each interval of its support are zero. For $x \in (t_j, t_{j+1})$ we have that $T_j^3(t_{j+1}) = \delta_j^3(r)\tilde{T}_r^3(t_{j+1})$. Since (3) applied to $\mathcal{S}(3, \tilde{\boldsymbol{m}}, d\boldsymbol{\sigma}^{(1)}, \Delta)$ implies that $\tilde{T}_r^3(t_{j+1}) = 1$, it remains to show that the middle coefficient equals 1. For $x \in (t_j, t_{j+1})$, property (3) again leads to

$$T_{j-1}^3(x) + T_j^3(x) + T_{j+1}^3(x) = 1.$$

We expand T_i^3, $i \in \{j-1, j, j+1\}$ in $\mathcal{S}(3, \tilde{\boldsymbol{m}}, d\boldsymbol{\sigma}^{(1)}, \Delta)$, and rearrange the terms to obtain

$$\begin{aligned}\tilde{T}_r^3(x)[T_{j-1}^3(t_{j+1}) + T_j^3(t_{j+1})] + \delta_j^3(r+1)\tilde{T}_{r+1}^3(x)& \\ + \tilde{T}_{r+2}^3(x)[T_j^3(t_{j+2}) + T_{j+2}^3(t_{j+2})] &= 1.\end{aligned}$$

The expressions in square brackets are equal to 1 by (3) applied at the knots t_{j+1}, t_{j+2}. But the partition of unity property must also hold for Chebyshev B-splines in $\mathcal{S}(3, \tilde{\boldsymbol{m}}, d\boldsymbol{\sigma}^{(1)}, \Delta)$, and, since the expansion of unity in this space must be unique, we have $\delta_j^3(r+1) = 1$. □

The more important "cubic" version follows from Theorem 2 and the derivative formula (4):

Theorem 3. *Let $T^4_{j,d\boldsymbol{\sigma}} \in \mathcal{S}(4, \boldsymbol{m}, d\boldsymbol{\sigma}, \Delta)$, $\tilde{T}^4_{j,d\boldsymbol{\sigma}} \in \mathcal{S}(4, \tilde{\boldsymbol{m}}, d\boldsymbol{\sigma}, \Delta)$, and let $\boldsymbol{m}$, $\tilde{\boldsymbol{m}}$ be multiplicity vectors as in Theorem 2. Then there exist positive $\delta_j^4(i)$, dependent on $d\sigma_3$, such that $T^4_{j,d\boldsymbol{\sigma}} = \sum_{i=r}^{r+3} \delta_j^4(i)\tilde{T}^4_{i,d\boldsymbol{\sigma}}$, where $r = r_j$ satisfies $t_j = \tilde{t}_{r_j} < \tilde{t}_{r_j+1}$. If the the extended partition is $\{t_1, \dots t_{k+8}\}$, and $\{\tilde{t}_1, \dots \tilde{t}_{2k+8}\}$ is the extended partition with double interior knots, then $\delta_j^4(i)$, $i = r, \dots r+3$ are explicitly determined by*

$$\delta_j^4(r) = \frac{T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+1})\tilde{C}(r)}{T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+1})\tilde{C}(r) + \tilde{C}(r+1) + T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+2})\tilde{C}(r+2)},$$

$$\delta_j^4(r+1) = \frac{T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+1})\tilde{C}(r) + \tilde{C}(r+1)}{T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+1})\tilde{C}(r) + \tilde{C}(r+1) + T^3_{j,d\boldsymbol{\sigma}^{(1)}}(t_{j+2})\tilde{C}(r+2)},$$

$$\delta_j^4(r+2) = \frac{T^3_{j+1,d\boldsymbol{\sigma}^{(1)}}(t_{j+3})\tilde{C}(r+4) + \tilde{C}(r+3)}{T^3_{j+1,d\boldsymbol{\sigma}^{(1)}}(t_{j+2})\tilde{C}(r+2) + \tilde{C}(r+3) + T^3_{j+1,d\boldsymbol{\sigma}^{(1)}}(t_{j+3})\tilde{C}(r+4)},$$

$$\delta_j^4(r+3) = \frac{T^3_{j+1,d\boldsymbol{\sigma}^{(1)}}(t_{j+3})\tilde{C}(r+4)}{T^3_{j+1,d\boldsymbol{\sigma}^{(1)}}(t_{j+2})\tilde{C}(r+2) + \tilde{C}(r+3) + T^3_{j+1,d\boldsymbol{\sigma}^{(1)}}(t_{j+3})\tilde{C}(r+4)},$$

where as in (5),

$$\tilde{C}(i) = \int_{\text{support}} \tilde{T}^3_{i,d\boldsymbol{\sigma}^{(1)}} d\sigma_2. \tag{7}$$

Proof: We expand $T^4_{j,d\boldsymbol{\sigma}}$ in terms of less smooth $\tilde{T}^4_{i,d\boldsymbol{\sigma}}$:

$$T^4_{j,d\boldsymbol{\sigma}}(x) = \sum_{i=r-2}^{r+6} \delta^4_j(i)\tilde{T}^4_{i,d\boldsymbol{\sigma}}(x),$$

and apply (4) to obtain

$$\frac{T^3_{j,d\boldsymbol{\sigma}}(x)}{C_3(j)} - \frac{T^3_{j+1,d\boldsymbol{\sigma}}(x)}{C_3(j+1)} = \sum_i \frac{\delta^4_j(i) - \delta^4_j(i-1)}{\tilde{C}_3(i)} \tilde{T}^3_{i,d\boldsymbol{\sigma}^{(1)}}(x).$$

We may then use Theorem 2 to expand $T^3_{j,d\boldsymbol{\sigma}}$, and write a linear system for δ^4_j, which has the above explicit solution. Details are omitted since the construction very much resembles the construction of ν–splines in [10]. □

It is not difficult to prove that δ^4_j are discrete Chebyshev splines, that is, they form a partition of unity, *i.e.*, $\sum_i \delta^4_i(j) = 1$ [10]; this also holds for general order splines.

We also note that coalescence of the knots yields an expression for a complete "Chebyshev cubic" spline on triple partitions.

§3. Examples

We investigate how much of the above theory can be applied to some known spaces of Chebyshev splines, and what special properties must be used to obtain the stability of the algorithm for the evaluation of Chebyshev B-splines.

Remark 4. Minimizing elastic energy of an inhomogenuous rod leads to a minimization problem for the functional

$$\int_a^b (\mathrm{E}(s)\mathrm{I}(s)u''(s))^2\, ds \to \min,$$

where E is Young's modulus of elasticity, and I is the moment of inertia of its cross section. We think of the rod as being made of pieces of different material (E piecewise constant), or different cross section (I is piecewise constant) on intervals $[x_i, x_{i+1})$ that partition $[a,b)$. In either case, the Euler equation is $L_{4,d\boldsymbol{\sigma}}u := (w\,u'')'' = 0$, where $d\boldsymbol{\sigma} := (dx, d\sigma, dx)^{\mathrm{T}})$, and $d\sigma$ is the measure generated by the piecewise constant density $w := 1/\mathrm{EI}, w|_{[x_i,x_{i+1}]} =: w_i$.

In CAGD such splines are known as ν–splines, introduced by Foley [2]. It follows from the variational formulation that ν–splines possess continuous derivatives, and that jumps in second derivatives must be continuous:

$$w_{i+1}u''(x_i) - w_i u''(x_i) = 0. \tag{8}$$

If the breakpoints for w are points of the partition Δ, it is not difficult to see (2) that (8) is equivalent to the continuity of the second generalized derivatives $L_{2,d\boldsymbol{\sigma}}$ across the knots. We may evaluate $T^4_{i,d\boldsymbol{\sigma}}$ by scalar products of positive, known quantities; for instance it follows (with the knots enumerated as in Theorem 3), that

$$T_2^4 = \frac{\gamma_2^3(4)}{\|\gamma_2^3\|}\tilde{B}_4^4 + \frac{\gamma_2^3(4)+\gamma_2^3(5)}{\|\gamma_2^3\|}\tilde{B}_5^4 + \frac{\gamma_3^3(7)+\gamma_3^3(8)}{\|\gamma_3^3\|}\tilde{B}_6^4 + \frac{\gamma_3^3(8)}{\|\gamma_3^3\|}\tilde{B}_7^4,$$

where

$$\gamma_2^3(4) = \frac{d\sigma(t_2,t_3)}{d\sigma(t_2,t_4)}(t_4-t_2), \quad \gamma_2^3(5) = t_4 - t_3,$$
$$\gamma_2^3(7) = t_5 - t_4, \quad \gamma_2^3(8) = \frac{d\sigma(t_5,t_6)}{d\sigma(t_4,t_6)}(t_6-t_4),$$
$$\|\gamma_l^3\| = \sum_i \gamma_l^3(i) \quad \text{for} \quad l = 2,3.$$

One should note that $\tilde{B}_i^4$ are ordinary polynomial splines (by raising the multiplicity of the knots we avoid the second derivative condition), and thus are readily calculated by the de Boor-Cox recurrence. This is an example of a Chebyshev system which is not an ECC-system, since $d\sigma$ is generated by a non-continuous density.

Remark 5. If $d\sigma$ in Remark 4 is a Lebesgue measure, the standard knot insertion formula for a "homogenuous rod" *i.e.*, cubic splines appears:

$$B_2^4 = \frac{t_3-t_2}{t_5-t_2}\tilde{B}_4^4 + \frac{t_4-t_2}{t_5-t_2}\tilde{B}_5^4 + \frac{t_6-t_4}{t_6-t_3}\tilde{B}_6^4 + \frac{t_6-t_4}{t_6-t_3}\tilde{B}_7^4.$$

Remark 6. In the last polynomial example, we consider the q-splines introduced by Kulkarni and Laurent [3], which are piecewisely linear combinations of the functions in the canonical system $\{1, u_2, u_3, u_4\}$:

$$u_2(x) = \int_a^x dt_2,$$
$$u_3(x) = \int_a^x dt_2 \int_a^{t_2} q(t_3)\,dt_3,$$
$$u_4(x) = \int_a^x dt_2 \int_a^{t_2} q(t_3)\,dt_3 \int_a^{t_3} dt_4,$$

where q is piecewise linear:

$$q(x)|_{[t_i,t_{i+1}]} := \frac{q_{i+1}-q_i}{h_i}(x-t_i) + q_i,$$

$h_i := t_{i+1} - t_i$, and $q_i > 0$. This may be thought of as an elastic rod with elastic properties changing in a reciprocally linear way; the more physically

x_0	x_1	x_2	x_3	x_4	x_5	x_6
$t_1\,\hat{t}_1$	t_5	t_6	t_7	t_8	t_9	$t_{10}\,\hat{t}_{20}$
$t_2\,\hat{t}_2$	$\hat{t}_5$	$\hat{t}_8$	$\hat{t}_{11}$	$\hat{t}_{14}$	$\hat{t}_{17}$	$t_{11}\,\hat{t}_{21}$
$t_3\,\hat{t}_3$	$\hat{t}_6$	$\hat{t}_9$	$\hat{t}_{12}$	$\hat{t}_{15}$	$\hat{t}_{18}$	$t_{12}\,\hat{t}_{22}$
$t_4\,\hat{t}_4$	$\hat{t}_7$	$\hat{t}_{10}$	$\hat{t}_{13}$	$\hat{t}_{16}$	$\hat{t}_{19}$	$t_{13}\,\hat{t}_{23}$

Fig. 2. Triple knots.

sound model in which material properties are changing linearly does not lead to "Chebyshev polynomial splines" [7], but involves logarithmic weights, and does not seem to be very useful.

Some parts of the construction can be made more explicit, *i.e.*, (7) can be integrated to obtain

$$\begin{aligned}\tilde{C}_3(r-1) &= \frac{h_i}{4}\left[\frac{2q_i}{2q_i+q_{i+1}} + \frac{2q_{i+1}}{q_i+2q_{i+1}}\right],\\ \tilde{C}_3(r) &= \frac{1}{4}\left[\frac{q_i h_i}{q_i+2q_{i+1}} + h_i + h_{i+1} + \frac{q_{i+2}h_{i+1}}{2q_{i+1}+q_{i+2}}\right].\end{aligned}$$

We can stably calculate "parabolic" B-splines required by Theorem 2 in two important points via

$$T_i^3(t_{i+1}) = \frac{h_i(q_i+2q_{i+1})}{6C_2(i)}, \quad T_i^3(t_{i+2}) = \frac{h_{i+2}(2q_{i+2}+q_{i+3})}{6C_2(i+1)},$$

where

$$C_2(i) = \int_{t_i}^{t_{i+2}} B_i^2(t)q(t)\,dt = \frac{1}{6}[(q_i+2q_{i+1})h_i + (2q_{i+1}+q_{i+2})h_{i+1}].$$

One can in fact express $\tilde{T}^4_{i,d\sigma}$ in terms of B-splines on a triple net (Bernstein polynomials). The enumeration of the knots is as in Fig. 2, with quadruple knots at both ends and triple knots $\hat{t}_i$ in the interior. If s is an index such that $t_i = \hat{t}_{s-2} = \hat{t}_{s-1} = \hat{t}_s < \hat{t}_{s+1}$, then

$$\begin{aligned}\tilde{T}^4_{r-1} = {}& \frac{1}{\tilde{C}_3(r-1)}\frac{2q_i}{2q_i+q_{i+1}}\frac{h_i}{4}\hat{B}^5_{s-2} + \frac{1}{\tilde{C}_3(r)}\left(\frac{h_i+h_{i+1}}{4}\right.\\ &\left. + \frac{q_{i+2}}{2q_{i+1}+q_{i+2}}\frac{h_{i+1}}{4}\right)\hat{B}^5_{s-1} + \frac{1}{\tilde{C}_3(r)}\frac{q_{i+2}}{2q_{i+1}+q_{i+2}}\frac{h_{i+1}}{4}\hat{B}^5_s,\end{aligned}$$

and an analogous result holds for $\tilde{T}^4_r$.

Remark 7. Let us now see how the theory may be applied to the "real" Chebyshev case by considering tension splines with uniform tension parameter that are piecewise in the null-space of the differential operator $D^2(D^2 - p^2)$,

where $p \geq 0$ is the tension parameter; see [4] for some more recent references. We may factor the differential operator as

$$D(D + p\tanh px)(D - p\tanh px)D = (\frac{1}{\cosh px}D)(\cosh^2 pxD)(\frac{1}{\cosh px}D)D,$$

and identify the measure vector $d\boldsymbol{\sigma} = (ds_2, \cosh ps_3 ds_3, \frac{ds_4}{\cosh^2 ps_4})^T$. Other measure vectors may be used, but this has the advantage that $L_{1,d\boldsymbol{\sigma}} \equiv D$ maps the tension spline space to the space of hyperbolic splines, where nice recurrencies exist. To apply Theorem 3, we need B-splines in the reduced system [14]:

$$T^3_{j,d\boldsymbol{\sigma}^{(1)}} = C \begin{cases} \frac{\sinh^2(p/2)(x-t_j)}{\sinh(p/2)(t_{j+2}-t_j)\sinh(p/2)(t_{j+1}-t_j)}, & x \in (t_j, t_{j+1}), \\ \frac{\sinh^2(p/2)(x-t_j)\sinh^2(p/2)(t_{j+2}-x)}{\sinh(p/2)(t_{j+2}-t_j)\sinh(p/2)(t_{j+2}-t_{j+1})} + & \\ \frac{\sinh^2(p/2)(t_{j+3}-x)\sinh^2(p/2)(x-t_{j+1}x)}{\sinh(p/2)(t_{j+3}-t_{j+1})\sinh(p/2)(t_{j+2}-t_{j+1})}, & x \in (t_{j+1}, t_{j+2}), \\ \frac{\sinh^2(p/2)(t_{j+3}-x)}{\sinh(p/2)(t_{j+3}-t_{j+1})\sinh(p/2)(t_{j+3}-t_{j+2})}, & x \in (t_{j+2}, t_{j+3}), \end{cases}$$

where the normalisation constant $C := \cosh\frac{p}{2}(t_{j+2} - t_{j+1})$ ensures the partition of unity (3). The problem of how to calculate $\tilde{T}^4_{i,d\boldsymbol{\sigma}}$ remains. If we use techniques like in Remark 6 to express tension splines on triple nets, we finally arrive at the numericaly nasty formula for a T-spline with support $[t_i, t_{i+1}]$, $(h := t_{i+1} - t_i)$, that is $\hat{T}^4_{i,d\boldsymbol{\sigma}}\frac{1}{\sinh\frac{p}{2}}$ is given by

$$\frac{(\sinh ph - ph)(\cosh p(x - t_i) - 1) - (\cosh ph - 1)(\sinh p(x - t_i) - p(x - t_i))}{2(\sinh ph - ph)(ph/2\cosh ph/2 - \sinh ph/2)}.$$

The above formula is a special case of the integrated version of the derivative formula (4). Taylor expansions may be used to calculate the above expression for a small p (approx. $p < 0.5$ in double precision, IEEE standard), and also an asymptotic formula for the ultimate almost linear spline, ideas similar to the ones used by Rentrop [8]. In the middle range it is best to utilize the generalized Taylor expansion from Lemma 1 in the vicinity of the inflexion point of $\hat{T}^4_{i,d\boldsymbol{\sigma}}$, which is defined as the solution of the equation $L_{3,d\boldsymbol{\sigma}}\hat{T}^4_{i,d\boldsymbol{\sigma}}(a_{\text{inflex}}) = 0$. On the standard interval $[0, 1]$, one obtains $a_{\text{inflex.}} = \frac{1}{2p}\log\frac{\exp(p)-p-1}{p+\exp(-p)-1}$; both limits never approach boundaries.

For $h = 1$ we can thus obtain an absolutely stable formula in the range $0.5 < p < 700$ with very few arithmetic operations. Translation invariance of splines in tension enables calculation on any interval. We also note that in order to have a complete algorithm, one must have a kind of Gaussian integration formula in closed form to calculate the normalisation constants (5).

Acknowledgments. This research was supported by Grant 037011, by the Ministry of Science and Technology of the Republic of Croatia.

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